# Infinite Reasoning

## Structura

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# 1 Truth

- Tarski:
  - T-convention: $\text{True}_\text{L}(\langle\phi\rangle) \leftrightarrow \phi$
  - ToF $\forall x(\text{Sent}_\text{L}(x) \rightarrow (\text{True}_\text{L}(x) \vee \text{True}_\text{L}(\dot{\neg}x)))$ is not guaranteed.
  - Infinite rule:

$$- \quad (\forall \text{Sent}_\text{L})\ \frac{\phi(a_1), \phi(a_2), ..., \phi(a_n), ...}{\forall x \text{Sent}_\text{L}(x) \rightarrow \phi(x)}$$

  - Two objections from Tarski:
    1. General doubts about infinite approaches: "It may well be doubted whether there is any place for the use of such a rule within the limits of the existing conception of the deductive method." [1]
    2. Consistency will not be proved. ($\omega$-inconsistent meta-theory)
- Horwich
  - Minimalist Approach towards Truth: T convention is all there is to say about the conception of Truth. (Against Tarski: sentences' truth conditions are further specified by their constituents)
  - A challenge: what explains ToF?
  - An explanatory premise: "Whenever someone is disposed to accept, for any proposition of type K, that it is G (and to do so for uniform reasons) then he will be disposed to accept that every K-proposition is G." [2]

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[1] Tarski, A. (1956). *Logic, semantics, metamathematics: Papers from 1923 to 1938*. Oxford: Oxford University Press, p.260.

[2] Horwich, P. (2005). "A minimalist critique of Tarski on truth". In J. C. Beall and B. Armour-Garb (eds.), *Deflationism and paradox*, Oxford: Oxford University Press. p. 84.

## 2 Arithmetic

- Robinson Arithmetic $Q$
  Q1 $\forall x \forall y(x \neq y \rightarrow Sx \neq y)$
  Q2 $\forall(x0 \neq Sx)$
  Q3 $\forall x(x \neq 0 \rightarrow \exists y(x = Sy))$
  Q4 $\forall x((x + 0) = x)$
  Q5 $\forall x \forall y(x + Sy = S(x + y))$
  Q6 $\forall x(x \times 0) = 0$
  Q7 $\forall x \forall y(x \times Sy) = ((x \times y) + x)$

- Peano Arithmetic $PA$
  Q1 $\forall x \forall y(x \neq y \rightarrow Sx \neq y)$
  Q2 $\forall x 0 \neq Sx$
  Q4 $\forall x(x + 0) = x$
  Q5 $\forall x \forall y(x + Sy = s(x + y))$
  Q6 $\forall x(x \times 0) = 0$
  Q7 $\forall x \forall y(x \times Sy) = ((x \times y) + x)$
  Induction Axiom: $\forall \bar{y}((\phi(0, \bar{y}) \wedge \forall x(\phi(x, \bar{y}) \rightarrow \phi(Sx, \bar{y}))) \rightarrow \forall x \phi(x, \bar{y}))$
  Second-order version: $\forall X((X0 \wedge \forall x(Xx \rightarrow XSx)) \rightarrow \forall y(Xy))$

- True Arithmetic $TA$
  TA = Th($\mathbb{N}$)

- Several facts

  1. TA is not recursively enumerable.
  2. PA and Q are not complete.
  3. Q is just what needed to capture each decidable property. ($\Sigma_1$-complete)

- $\omega$-rule

$$(\omega)\ \frac{\phi(0), \phi(S0), \phi(SS0)...}{\forall x(\mathbf{N}(x) \rightarrow \phi(x))}$$

  - Hilbert: "If it has been proved, that every time z is a given numeral, the formula A(z) becomes a correct numerical formula, then (x)A(x) may be used as a first formula [in a derivation, i.e., as an axiom]". [3]
  - $\omega_H$-rule: If $\forall n \in \mathbf{N}[\phi(\bar{n})$ is numerically correct], then $\vdash \forall x \phi(x)$.

[3] Buldt, B. (2004). "On RC 102 43-14". In S. Awodey and C. Klein (Eds.), *Carnap Brought Home: The View from Jena*. Chicago and LaSalle: Open Court, p.228

- $\omega$-rule closes Q and PA to be complete.
- $\omega$-rule is strictly stronger than induction axiom.

# 3 Objections

- Supertask Argument
  - Argument:
    1. If we could engage in infinite reasoning, then we would be able to construct infinitely long proofs.
    2. We are not able to construct infinitely long proofs.
    3. We can't engage in infinite reasoning.
  - Objection: Reasoning doesn't always require the construction of formal proofs. (dogs, children)
  - Computational Argument
    * Argument:
      1. If we could engage in infinite reasoning, then we would be able to enumerate TA.
      2. We are not able to enumerate TA.
      3. We can't engage in infinite reasoning.
    * Church Thesis: A function of positive integers is effectively calculable only if lambda-definable (or, equivalently, recursive).
    * Objection:
      1. It requires too much that one always infer the conclusion upon accepting the premises. (One can make mistakes)
      2. One may well not accept all premises: one can not be presented with and make a prior decision on all of the infinite premises. (tension with dispositionalsim)
  - Inference Argument
    * Argument:
      1. If we could engage in infinite reasoning, then we would be able to perform infinite inferences.
      2. We are not able to perform infinite inferences.
      3. We can't engage in infinite reasoning.
    * The case for infinite reasoning and the case for an ability to follow the omega rule both stand or fall with the case for infinite inferences.

# 4 The Nature of Inference

- Functionalism: a causal state transition between accepting all of $q_1, ..., q_n$ to accepting $p$ is an inference just in case the transition plays the inference role.
- Inferential Role: the casual transition between Sally's premises attitudes and her conclusion attitudes plays the inference role just in case:
  1. Counterfactual: If Sally were asked for her reasons for here conclusion attitude, she would cite the content of her premise attitudes.
  2. Sally's broader mental state is properly aligned –she isn't disposed to judge inferences from these premises to that conclusion to be faulty.
  3. Sally can exercise some measure of control over the transition from her premise attitudes to her conclusion attitude.

# 5 Is Infinite Inference Possible?

- Is it possible to accept infinitely many premises?
  - Dispositionalists and Interpretationalists will allow infinite beliefs.
  - Even representationalists will also recognize the need for dispositional account of some beliefs and hence allow infinite (dispositional) beliefs.
- Is transition involving state of accepting infinite premises possible?
  - Objection: Casual Compactness. No a priori reason to reject the state of accepting infinitely many premises participating in transition, if one already concede that one can be in such a state.
  - Objection: Implicit beliefs. To say that one implicitly believes all of infinitely many premises is just to say that she is in some particular mental state.
- Is counterfactual possible?
  - One will not have a disposition to finish citing all of infinitely many premises. But one do have the ability to cite premise(n+1) *given* premise(n). Implicitly beliefs are potentially explicit only individually not jointly.
  - A further limitation: the premises are themselves enumerable by the agent. (consider a rule with a premise for a numeral n only if n is the Godel Number of an arithmetical truth given a fixed encoding)
- Is Alignment possible?

- it is satisfied trivially: if one can't make judgement about the inference, then one surely can't judge the inference a bad one.
- Demonstratives or descriptions are available.

- Is Control possible?
  - It is not required that our control is perfect. Quasi-control will be enough.
  - Also, we have set restriction that premises should be enumerable by agents.

# 6 An Example

- Goal: Describe a case where infinite reasoning is a natural interpretation, so that infinite reasoning is not only possible but actual.
- Supertask computer: a computer able to perform a countably infinite number of computations in a finite time.
- Goldbach's conjecture checked: The supertask computer check Goldbach property for 0 in half a minute, 1 in half of half a minute... If Goldbach property does not hold for some number, the computer halts.
- Objections
  - Computational argument reinstated. Supertask computations are not recursive. We only accept the results given and use some bit of omega reasoning.
  - Reinterpretation as Induction.

$$\text{(GB Induction)}\ \frac{GB(0) \qquad \forall x(GB(x) \rightarrow GB(Sx))}{\forall x GB(x)}$$

    However 1)it is unnatural 2) it may not avoid omega reasoning for we need omega reasoning to establish $\forall x(GB(x) \rightarrow GB(Sx))$.
  - There is no conceptual gap between accepting premises here and its conclusion. However, consider a Martian society: M = PA + $\neg\forall x GB(x)$. If the machine does not halt, at most it shows that M is $\omega$-inconsistent, but not inconsistent.
  - A further objection: (UR) $\forall x(Prov_M(\langle GB(x)\rangle) \rightarrow GB(x))$. However, UR is not provable from PA, and there is no reason for Martians to accept it.
  - A possible interpretation: MT. If the computation halts, then there is an x such that x does not have Goldbach property.