StePHan Krämer: Truthmaker Equivalence

Yudi Huang

中国人民大學哲學院
SCHOOL OF PHILOSOPHY, RENMIN UNIVERSITY OF CHINA

April 18th

Contents

Introduction

Hyperintensionality

See §2 of the first talk.

Granularity Problem

How fine-grained are hyperintension*s*? And Why?

Fine-grained	Syntactic Identity	$A \approx A$
↑	?	
Coarse-grained	Necessary Equivalence	$A \approx (A \wedge B) \vee (A \wedge \neg B)$...

Granularity Problem

How fine-grained are hyperintension*s*? And Why?

Fine-grained	Syntactic Identity	$A \approx A$
↑	?	
Coarse-grained	Necessary Equivalence	$A \approx (A \wedge B) \vee (A \wedge \neg B)$...

The answer to these questions may vary with the subject matter we are looking at.

Example 1: Grounding

Correia (2020):

> *Two facts x and y are ground-theoretically equivalent iff they play the same ground theoretic role...*

Example 1: Grounding

Correia (2020):

> *Two facts x and y are ground-theoretically equivalent iff they play the same ground theoretic role...*

Krämer (2019):$A \not\approx_G \neg\neg A$, since $A < \neg\neg A$ but $\neg\neg A \not< \neg\neg A$ (irreflexivity).

Example 1: Grounding

Correia (2020):

> *Two facts x and y are ground-theoretically equivalent iff they play the same ground theoretic role...*

Krämer (2019):$A \not\approx_G \neg\neg A$, since $A < \neg\neg A$ but $\neg\neg A \not< \neg\neg A$ (irreflexivity).

NOTE: 1. Krämer (2019) also rejects some highly plausible equivalence such as $(A \vee B) \vee C \approx_G A \vee (B \vee C)$; 2.Grounding theorists have embraced (or at least proposed) different logic of ground-theoretic equivalence, which validate different equivalences. The acceptability of these equivalences are controversial: $A \approx_G A \wedge A, A \wedge (B \vee C) \approx_G (A \wedge B) \vee (A \wedge C)$... (For a comprehensive review, see Correia (2020))

Example 2: Counterfactuals

Some have argued hyperintensionality of counterfactual context in favor of a rule called Simplification of Disjunctive Antecedent (SDA), which, together with the rule of Substitution of Classical Equivalence, will trivialize the logic of counterfactuals in the sense that counterfactuals collapse into strict conditionals.

Example 2: Counterfactuals

Nute (1980): Not all substitutions are to be abandoned, but which are not? He listed ten acceptable rules of substitution, which could be seen as 'counterfactual equivalence':

1. $A \vee B \approx_{\Box\!\!\rightarrow} B \vee A$ (ST1)
2. $A \vee (B \vee C) \approx_{\Box\!\!\rightarrow} (A \vee B) \vee C$ (ST2)
3. $A \vee A \approx_{\Box\!\!\rightarrow} A$ (ST3)
4. $A \wedge B \approx_{\Box\!\!\rightarrow} B \wedge A$ (ST4)
5. $A \wedge (B \wedge C) \approx_{\Box\!\!\rightarrow} (A \wedge B) \wedge C$ (ST5)
6. $\neg(A \wedge B) \approx_{\Box\!\!\rightarrow} \neg A \vee \neg B$ (ST6)
7. $\neg(A \vee B) \approx_{\Box\!\!\rightarrow} \neg A \wedge \neg B$ (ST7)
8. $A \wedge B \approx_{\Box\!\!\rightarrow} \neg(\neg A \vee \neg B)$ (ST8)
9. $A \wedge (B \vee C) \approx_{\Box\!\!\rightarrow} (A \wedge B) \vee (A \wedge C)$ (ST9)
10. $A \approx_{\Box\!\!\rightarrow} \neg\neg A$ (ST10)

Example 2: Counterfactuals

Abandoned instances of classical equivalence are $A \vee (B \wedge C) \leftrightarrow (A \vee B) \wedge (A \vee C)$ (Distribution of $\vee$ over $\wedge$) and $A \leftrightarrow A \wedge A$ (Collapse of $\wedge$).

Nute (1980) proves that (SDA)+(ST1-10)+(Distribution $\vee/\wedge$) will trivialize the logic of counterfactuals.

郭顺利 (2012) proves that (SDA)+(ST1-10)+(Collapse $\wedge$)+(Classical Closure of Conditional Consequent) imply (Distribution $\vee/\wedge$) and hence trivialize the logic of counterfactuals.

A Non-metaphysical Example: Imperatives and Deontics

Fine (2018b,a): $A \vee B$ is not equivalent to $A \vee B \vee (A \wedge B)$ in imperatives or deontic contexts. (This means $A \wedge B$ does not entail $A \vee B$ in these contexts).

A Non-metaphysical Example: Imperatives and Deontics

Fine (2018b,a): $A \vee B$ is not equivalent to $A \vee B \vee (A \wedge B)$ in imperatives or deontic contexts. (This means $A \wedge B$ does not entail $A \vee B$ in these contexts).

By **shutting the door and closing the window**, one may fail to comply with the imperative *to shut the door or close the window*.

The permission *to have ice-cream or chocolate* is not automatically a permission *to have either or both*.

Remarks

First, the examples reveal that the fineness of grain of propositional contents may differ in different contexts. That's why I use the plural form hyperintension*s* rather than its singular form. To solve the problem of granularity is not only to determine which contents are equivalent, but also to specify the context in which they are so discriminated and explain why.

Truthmaker Semantics

Truthmaker Semantics

	Truthmaker Semantics	Possible World Semantics
Logical Space	States	Possible Worlds
	can be incomplete and/or inconsistent	complete and consistent
Truthmaking	Exact Verification	Necessitation
	fully relevant	not fully relevant
	non-monotonic	monotonic

表: Differences between TS and PWS

Exact Verification

Two characteristics of exact verification:

First, a state is *fully relevant* to the truth of the proposition it exactly verifies: the state of **it being sunny** exactly verifies the proposition that *it is sunny*, but does not verify that *it is windy or not windy*, since it has nothing to do with its being windy or not.

Exact Verification

Two characteristics of exact verification:

First, a state is *fully relevant* to the truth of the proposition it exactly verifies: the state of **it being sunny** exactly verifies the proposition that *it is sunny*, but does not verify that *it is windy or not windy*, since it has nothing to do with its being windy or not.

Second, exact verification is non-monontonic: the state of **it being sunny** exactly verifies the proposition that *it is sunny*, while the state of **it being sunny and hot** does not, even if the latter contains the former.

State Space

Definition (State Space)

*A **state space** is an ordered pair $\langle S, \sqsubseteq \rangle$, where S (states) is a non-empty set and $\sqsubseteq$ (parthood/substate relation) is a complete partial order on S. (A order is complete iff each subset of its domain has a least upper bound w.r.t. the order.)*

State Space

Definition (State Space)

*A **state space** is an ordered pair $\langle S, \sqsubseteq \rangle$, where S (states) is a non-empty set and $\sqsubseteq$ (parthood/substate relation) is a complete partial order on S. (A order is complete iff each subset of its domain has a least upper bound w.r.t. the order.)*

Definition (Fusion)

*Given a subset $T = \{t_1, t_2, ..., t_n\}$ of S, we call the least upper bound of T the **fusion** of T. We use $\sqcup T$ or more often $t_1 \sqcup t_2 \sqcup ... \sqcup t_n$ to represent the fusion of T.*

Truthmaker Content

Definition (Unilateral Proposition)

A ***unilateral proposition*** *on a state space* $\langle S, \sqsubseteq \rangle$ *is a subset of* S.

Definition (c-inclusive/f-inclusive/regular verification)

1. s ***c-inclusively*** *verifies* P *iff* $t \sqsubseteq s \sqsubseteq u$ *for some* $t, u \in P$. *We use* P^C *(the* ***c-inclusive content*** *of* P*) to denote the set of c-inclusive verifiers of* P.
2. s ***f-inclusively*** *verifies* P *iff for some* $S \subseteq P$, $s = \bigsqcup S$. *We use* P^F *for the* ***f-inclusive content*** *of* P.
3. s ***regularly*** *verifies* P *iff* $t \sqsubseteq s \sqsubseteq \bigsqcup P$ *for some* $t \in P$. *We use* P^R *for the* ***regular content*** *of* P.

Conjunction and Disjunction

Definition

For unilateral propositions P, Q on some state space,

- $P \wedge Q = \{s \sqcup t : s \in P \ \& \ t \in Q\}$
- $P \vee Q = P \cup Q$

That is to say:

- A state s verifies $P \wedge Q$ iff $s = t \sqcup u$ for some $t \in P$ and some $u \in Q$;
- A state s verifies $P \vee Q$ iff s verifies P or s verifies Q.

Bilateral Proposition and Negation

Definition (Bilateral Proposition)

A ***bilateral proposition*** $\mathbf{P}$ *on a state space* $\langle S, \sqsubseteq \rangle$ *is a pair of unilateral propositions on* $\langle S, \sqsubseteq \rangle$*. If* $\mathbf{P}$ *is a bilateral proposition, its first (second) coordinate will be denoted as* $\mathbf{P}^+ (\mathbf{P}^-)$ *and be called the positive (negative) content of* $\mathbf{P}$*.*

Definition

For bilateral propositions $\mathbf{P}, \mathbf{Q}$ *on some state space,*

- $\mathbf{P} \wedge \mathbf{Q} = (\mathbf{P}^+ \wedge \mathbf{Q}^+, \mathbf{P}^- \vee \mathbf{Q}^-)$
- $\mathbf{P} \vee \mathbf{Q} = (\mathbf{P}^+ \vee \mathbf{Q}^+, \mathbf{P}^- \wedge \mathbf{Q}^-)$
- $\neg \mathbf{P} = (\mathbf{P}^-, \mathbf{P}^+)$

Varieties of Truthmaker Equivalence

As we can see in the Introduction part, hyperintensional contents may have different granularities in different contexts. This paper is to distinguish a variety of equivalence relations between propositions within the framework of truthmaker semantics. I also believe that this distinction can shed light on the question why certain hyperintensional context is so and so discriminated.

Varieties of Truthmaker Equivalence

Definition

For unilateral propositions P and Q, say that

- P *is exactly equivalent to* Q ($P \approx_E Q$) *iff* $P = Q$
- P *is c-equivalent to* Q ($P \approx_C Q$) *iff* $P^C = Q^C$
- P *is f-equivalent to* Q ($P \approx_F Q$) *iff* $P^F = Q^F$
- P *is r-equivalent to* Q ($P \approx_R Q$) *iff* $P^R = Q^R$

Truthmaker Implication

Proposition

For unilateral propositions P and Q:

1. $P \subseteq Q$ iff $P \vee Q \approx_E Q$
2. $P^C \subseteq Q^C$ iff $P \vee Q \approx_C Q$
3. $P^F \subseteq Q^F$ iff $P \vee Q \approx_F Q$
4. $P^R \subseteq Q^R$ iff $P \vee Q \approx_R Q$

证明.

Omitted. □

Deductive Systems

Syntax

Let $\mathcal{L}$ be a (countable) standard propositional language with connectives $\wedge, \vee, \neg$. We will use the *language of equivalence* $\mathcal{L}_{\approx}$ consisting of all $\mathcal{L}$-equivalences, expressions of the form $A \approx B$ with A, B $\mathcal{L}$-formulas.

Axioms of the Systems

<table>
<tr><th></th><th>$\mathfrak{D}_b$</th><th>$\mathfrak{D}_e$</th><th>$\mathfrak{D}_f$</th><th>$\mathfrak{D}_c$</th><th>$\mathfrak{D}_r$</th></tr>
<tr><td>$A \vee A \approx A$</td><td colspan="5">✓</td></tr>
<tr><td>$A \vee B \approx B \vee A$</td><td colspan="5">✓</td></tr>
<tr><td>$A \vee (B \vee C) \approx (A \vee B) \vee C$</td><td colspan="5">✓</td></tr>
<tr><td>$A \wedge B \approx B \wedge A$</td><td colspan="5">✓</td></tr>
<tr><td>$A \wedge (B \wedge C) \approx (A \wedge B) \wedge C$</td><td colspan="5">✓</td></tr>
<tr><td>$\neg\neg A \approx A$</td><td colspan="5">✓</td></tr>
<tr><td>$\neg(A \vee B) \approx \neg A \wedge \neg B$</td><td colspan="5">✓</td></tr>
<tr><td>$\neg(A \wedge B) \approx \neg A \vee \neg B$</td><td colspan="5">✓</td></tr>
<tr><td>$A \wedge (B \vee C) \approx (A \wedge B) \vee (A \wedge C)$</td><td colspan="5">✓</td></tr>
<tr><td>$A \rightarrow A \wedge A$</td><td></td><td colspan="4">✓</td></tr>
<tr><td>$A \wedge A \rightarrow A$</td><td></td><td></td><td>✓</td><td></td><td>✓</td></tr>
<tr><td>$(A \wedge B) \rightarrow A \vee (A \wedge (B \wedge C))$</td><td></td><td></td><td></td><td>✓</td><td>✓</td></tr>
</table>

表: Axioms

Rules

- $A \approx B / A \vee C \approx B \vee C$ (Preservation ($\vee$))
- $A \approx B / A \wedge C \approx B \wedge C$ (Preservation ($\wedge$))
- $A \approx B / B \approx A$ (Symmetry)
- $A \approx B, B \approx C / A \approx C$ (Transitivity)

表: Rules

Fine (2016) shows that the following rule of Positive Replacement is derivable in $\mathfrak{D}_b$ (and hence extended systems of it):

- $A \approx B / C \approx C(A/B)$, if $C(A/B)$ is the resultof replacing the occurrences of A in C by B, and no occurrences of A in C are in the scope of $\neg$. (PR)

Some Facts

Proposition

1. $\vdash_{f(r)} A \wedge A \approx A$ *Collapse* $(\wedge)$
2. $\vdash_{f(r)} A \vee B \approx A \vee B \vee (A \wedge B)$
3. $\vdash_{r} A \vee (B \wedge C) \approx (A \vee B) \wedge (A \vee C)$

Semantics

Definition (Model)

An $\mathcal{L}$-model is a triple $\mathfrak{M} = \langle S, \sqsubseteq, |\cdot|\rangle$ such that $\langle S, \sqsubseteq\rangle$ is a state space, and $|\cdot|$ maps every atomic sentence to a bilateral proposition on $\langle S, \sqsubseteq\rangle$. Given an $\mathcal{L}$-model, we can extend $|\cdot|$ to the complex formulas of $\mathcal{L}$:

- $|\neg A| = \neg|A|$
- $|A \wedge B| = |A| \wedge |B|$
- $|A \vee B| = |A| \vee |B|$

Semantics

Definition (Contents)

Given an $\mathcal{L}$-model $\mathfrak{M} = \langle S, \sqsubseteq, |\cdot|\rangle$, for any $\mathcal{L}$-formula A, the

- **exact** *content* $|A|_e$ *of* A *is* $|A|$
- **c – inclusive** *content* $|A|_c$ *of* A *is* $((|A|^+)^C, (|A|^-)^C)$
- **f – inclusive** *content* $|A|_f$ *of* A *is* $((|A|^+)^F, (|A|^-)^F)$
- **regular** *content* $|A|_r$ *of* A *is* $((|A|^+)^R, (|A|^-)^R)$

Semantics

Definition (Truth in a Model)

Given an $\mathcal{L}$-model $\mathfrak{M} = \langle S, \sqsubseteq, |\cdot| \rangle$, an $\mathcal{L}$-equivalence $A \approx B$ is **true** *in $\mathfrak{M}$ under the*

- **exact** *interpretation of $\approx$ iff $|A|^+ \approx_E |B|^+$, that is, $|A|^+ = |B|^+$*
- $\mathbf{c}$ **– inclusive** *interpretation of $\approx$ iff $|A|^+ \approx_C |B|^+$, that is, $(|A|^+)^C = (|B|^+)^C$*
- $\mathbf{f}$ **– inclusive** *interpretation of $\approx$ iff $|A|^+ \approx_F |B|^+$, that is, $(|A|^+)^F = (|B|^+)^F$*
- **regular** *interpretation of $\approx$ iff $|A|^+ \approx_R |B|^+$, that is, $(|A|^+)^R = (|B|^+)^R$*

Definition (Validity)

$A \approx B$ is **valid** *under an interpretation of $\approx$ iff for every model $\mathfrak{M}$, $A \approx B$ is true in $\mathfrak{M}$ under that interpretation of $\approx$. We write $\models_{e(c,f,r)} A \approx B$ to say that $A \approx B$ is valid under the corresponding interpretation.*

Soundness

Proposition

For all $\mathcal{L}$-equivalence $A \approx B$:

$$\vdash_{e/c/f/r} A \approx B \Rightarrow \models_{e/c/f/r} A \approx B$$

证明.

The proof is routine. □

Independence

Independence results are of (maybe only of) technical interests. The basic idea is this: to show that an axiom A is independence of a logic L is to show that L is sound w.r.t some class of structures but A is not valid in this class.

A point worth noting is that $A \to A \wedge A$ is independent of $\mathfrak{D}_b$, which, together with that $\mathfrak{D}_e$ is sound w.r.t. the class of all $\mathcal{L}$-model, shows that $\mathfrak{D}_b$ is not the logic of exact equivalence (while Fine (2018b) claims that it is).

Completeness

Overview

To prove the completeness:

$$\models A \approx B \Rightarrow \vdash A \approx B$$

It suffices to prove the cotraposition:

$$\nvdash A \approx B \Rightarrow \nvDash A \approx B$$

Now we find DNFs that are the 'labels' of the classes of provable equivalences and then only need to prove (let A^*, B^* be the disjunctive normal forms of A and B):

$$A^* \neq B^* \Rightarrow \nvDash A \approx B$$

So the disjunctive normal forms should have the property that whenever they are distinct, it is witnessed by some canonical model.

Preliminary

Some terminologies:

- Literal: $p_1, \neg p_1, \neg p_2 ...$
- Conjunctive form: $p_1 \wedge p_2 \wedge \neg p_2 \wedge p_7 \wedge ... \wedge p_{6251}$
- Disjunctive form: $(p_1 \wedge \neg p_2) \vee (p_1 \wedge p_{322}) \vee ... \vee (\neg p_{1024} \wedge p_{2048} \wedge p_{4096})$

Preliminary

More terminologies:

- Multi-set: $|x, x, x, y, y, z|$. No ordered. Elements can occur several times.
- $\#(S, x) = 3$ is the number of occurrences of x in S
- *Sum* of multi sets: $S + T$ is the multi-set U such that $\#(U, x) = \#(S, x) + \#(T, x)$ for all x

Preliminary

Given a conjunvtive form A, let $lit(A)$ be the multi-set of literals that includes each literal exactly as many times as it occurs as a conjunct in A.

Fact: for conjunctive forms A and B, if $lit(A) = lit(B)$ then $\vdash A \approx B$.

Standardization: Let function cf map each multi-set X of literals to a conjuctive form $cf(X)$ with $lit(cf(X)) = X$. Re-order the liberals so that each multi-set has only one 'label'.

Preliminary

Given a disjunctive form A, let $v(A)$ be $\{lit(B) : B \text{ is a disjunct in } A\}$.

Fact: for disjunctive forms A and B, if $v(A) = v(B)$ then $\vdash A \approx B$.

Standardlization: let function df map each set M of multi-sets of literals to some disjunctive form $df(M)$ such that:

1. $v(df(M)) = M$
2. no disjunct in $df(M)$ occurs more than once
3. all disjuncts in $df(M)$ are standard conjunctive forms.

Normalization

Lemma

Every formula is provably equivalent (in each system) to a disjunctive form.

证明.

In standard way. See 刘壮虎《逻辑演算》第六章第 2 节或徐明《符号逻辑讲义》§6.6. Note that we only need to use axioms and rules in $\mathfrak{D}_b$ (De Morgan Laws, Double Negation Elimination, Distribution of $\wedge$ over $\vee$ and PR, Symmetry and Transitivity). □

Standardization

Say that A and B are *strictly equivalent* within a given system iff A and B are provably equivalent within that system and $v(A) = v(B)$.

Lemma

Every disjunctive form is strictly equivalent to a standard disjunctive form.

证明.

Use PR implicitly. Use Commutativity ($\wedge$) and Associativity ($\wedge$) to standardize each conjunct. Use Commutativity ($\vee$) and Associativity ($\vee$) to re-order the disjuncts and use Collapse ($\vee$) to reduce repeated ones. □

Normal Form for Exact Truthmaking

Standard disjunctive forms do not suffice to prove the completeness for $\mathfrak{D}_e$. The following eample shows that some distinct SDFs have the same exact truthmakers:

Consider $A \wedge A$ and $A \vee (A \wedge A)$ with A a literal. Note that $\vdash_e A \vee (A \wedge A) \approx A \wedge A$ (this is exactly the axiom $A \rightarrow A \wedge A$). Let B and C be the SDF $df(v(A \wedge A))$ and $df(v(A \vee (A \wedge A)))$. Since $v(B) = v(A \wedge A)$ and $v(C) = (A \vee (A \wedge A))$, B and C are provably equivalent to $A \wedge A$ and $A \vee (A \wedge A)$ respectively, and hence are provably equivalent. By soundness, they have the same exact truthmakers in all models.

However, B and C are distinct SDFs, since $v(B) = v(A \wedge A) = \{|A, A|\} \neq \{|A|, |A, A|\} = v(A \vee (A \wedge A)) = v(C)$.

The point is, exact interpretation of $\approx$ does not dinstinguish between formulas like $A \wedge A$ and $A \vee (A \wedge A)$ (captured by the axiom $A \to A \wedge A$), also not between $A \vee (A \wedge A \wedge A \wedge A)$ and $A \vee (A \wedge A) \vee (A \wedge A \wedge A) \vee (A \wedge A \wedge A \wedge A)$

To see why, recall that each exact truthmaker of $A \wedge A$ is an exact truthmaker of $A \vee (A \wedge A)$, since the former is a disjunct of the latter. Moreover, each exact truthmaker of A is an exact truthmaker of $A \wedge A$. More complicated cases are similar.

Therefore, for each 'imperfect' disjunctive form, we add to it all the 'missing' disjuncts.

Definition (Subsuming)

For multi-sets of literals X and Y, say that

- *X is subsumed by Y iff (i) the same set underlies X and Y, and (ii) $\#(X, x) \leq \#(Y, x)$ for all x;*
- *X is properly subsumed by Y iff X is subsumed by Y and Y is not subsumed by X.*

Say that a conjunction form A is (properly) subsumed by a conjunctive form B iff $lit(A)$ is (properly) subsumed by $lit(B)$.

Definition (Subsuming)

For multi-sets of literals X and Y, say that

- *X is subsumed by Y iff (i) the same set underlies X and Y, and (ii) $\#(X, x) \leq \#(Y, x)$ for all x;*
- *X is properly subsumed by Y iff X is subsumed by Y and Y is not subsumed by X.*

Say that a conjunction form A is (properly) subsumed by a conjunctive form B iff $lit(A)$ is (properly) subsumed by $lit(B)$.

Example

- $|p, q, q|$ is subsumed by itself and properly by $|p, p, q, q, q|$...
- $p \wedge p \wedge q$ is (properly) subsumed by $p \wedge p \wedge p \wedge q \wedge q$...

Definition (Full Set)

A set M of multi-sets of literals is full iff every multi-set subsumed by some member of M is itself a member of M.
Say that disjunctive form A is full iff $v(A)$ is full.

Definition (Full Set)

A set M of multi-sets of literals is full iff every multi-set subsumed by some member of M is itself a member of M.
Say that disjunctive form A is full iff $v(A)$ is full.

Example

- $\{|p, p, q\}$ is not full; $\{|p|, |p, p|, |q|, |p, q|, |p, p, q|\}$ is full;
- $p \vee (p \wedge p \wedge q)$ is not full; $p \vee q \vee (p \wedge p) \vee (p \wedge q) \vee (p \wedge p \wedge q)$ is full.

Now we prove that every disjunctive form can be extended to a full disjunctive form in $\mathfrak{D}_e$.

Lemma

Every disjunctive form is provably equivalent within $\mathfrak{D}_e$ to a full disjunctive form.

证明.

We process step by step. Define F' fall-off value n as the number of multisets of literals X such that: (i) X is properly subsumed by some member of $v(F)$, and (ii) $X \notin v(F)$. When $n = 0$ then F is full and the process completes.

Suppose $n > 0$. Pick some multi-set of literals X which is not a member of $v(F)$ but is *immediately* subsumed by some $Y \in v(F)$, i.e., there is exacly one literal in Y that occurs less often in X, and it occurs exactly one time less. We show that $cf(X)$ can be added to F.

证明.

Let B be the disjunct of F with $lit(B) = Y$. Then B and $cf(X)$ are respectively provably equivalent to some conjunctive forms of the forms $(C \wedge C) \wedge D$ and $C \wedge D$. But

$$
\begin{aligned}
B &\approx (C \wedge C) \wedge D \\
&\approx (C \vee (C \wedge C)) \wedge D && \text{(Ecollapse}(\wedge)\text{,Preservation}(\wedge)) \\
&\approx (C \wedge D) \vee ((C \wedge C) \wedge D) && \text{(Distributivity}(\wedge/\vee)) \\
&\approx B \vee cf(X) && \text{(PR)}
\end{aligned}
$$

Replacing B in F by $B \vee cf(X)$, we get F with a fall-off value $< n$. □

A worry is that, the nice shape of standardized disjunctive form will be disordered. The following lemma shows that this is unfounded.

Lemma

Evey standard disjunctive form is provably equivalent within $\mathfrak{D}_e$ to a full standard disjunctive form.

证明.

Let F be a standard disjunctive form. By the previous lemma, F is provably equivalent to a full disjunctive form F'. By lemma 9, F' is strictly equivalent to some standard disjunctive form F^*. Since they are strictly equivalent, $v(F') = v(F^*)$ and so since F' is full, so is F^*. □

Canonical Models

Now we turn to the last step of completeness proof. Recall that for each pair of distinct disjunctive normal form, we need to construct a canonical model to witness this distinction.

The simplest strategy for defining canonical models for DNF does not suffice to prove completeness for exact truthmaking. See the beginning of 6.1 of the paper.

Canonical Model

Definition (Exact Canonical Model)

The **exact canonical model** $\mathfrak{M}_E$ *is the triple* $\langle S, \sqsubseteq, |\cdot| \rangle$ *with*

- $S = \mathcal{P}(\{(A, i) : A \text{ is a literal of } \mathcal{L} \text{ and } i \in \mathbb{N}\})$
- $\sqsubseteq$ *is the restriction of subsethood to* S
- $|p| = (\{\{(p, i)\} : i \in \mathbb{N}\}, \{\{(\neg p, i)\} : i \in \mathbb{N}\})$

Note that a exact verifier of p is of the form $\{(p, i)\}$ (a set) rather than (p, i) (an ordered-pair).

It is easy to verify that $\mathfrak{M}_E$ is indeed a model.

Lemma

In $\mathfrak{M}_E$, for any conjunctive form A and literal q, if q occurs a total of n times as a conjunct in A, then

1. *every verifier of A has at most n pairs of the form (q, i) as members;*
2. *some verifier of A has n distinct pairs of the form (q, i) as members;*
3. *every verifier of A has at least one pair of the form (q, i) as member, provided $n > 0$.*

证明.

Part (3) is immediate from definitions.

Prove (1) and (2) by induction on n. Case $n = 0$: (1) follows immediately from definitions, and (2) follows from (1) and model constuction.

证明.

Case $n = m+1$: Assume that A is of the form $B \wedge q$. Then B only contains n occurrences of q. By IH, (1) every verifier of B contains at most m pairs of the form (q, i) as members, and some verifier of B has m distinct pairs of the form (q, i) as members.

For (1), suppose s verifies A. Then $s = t \cup u$ for some t verifying B and some u verifying q. By constuction of canonical model, $u = \{(q, j)\}$ for some j. Since t contains at most m pairs of the form (q, i), s contains at most $m + 1 = n$ pairs of that form.

For (2), by IH, we can pick a verifier t of B with m distinct pairs of the form (q, i). Let $j = max\{i : (q, i) \in t\}$. Then $\{(q, j + 1)\}$ verifies q, and hence $s = t \cup \{(q, j + 1)\}$ verifies A and s contains $m + 1 = n$ pairs of the form (q, i). □

Lemma

Let A be a conjunctive form. In $\mathfrak{M}_E$, there is some verifier s of A such that every conjunctive form verified by s subsumes A.

证明.

Let s be a verifier of A as per (2) of the previous lemma. So for any literal q occuring $n \geq 1$ times in A, s has n distinct pairs of the form (q, i) as members. Suppose s verifies a conjunctive form B. Then by (1) of the previous lemma, each literal occuring in A occurs at least as often in B (other wise the verifier has n pairs but the literal occurs less than n times).

It remains to show that the same set underlies $lit(A)$ and $lit(B)$. We need to show every literal occurring in B occurs at least once in A. Suppose q occurs in B. Then by (3) of the previous lemma, s contains at least one pair of the form (q, i) as a member. Then by (1), q occurs in A. □

Proposition

In $\mathfrak{M}_E$, if A and B are distinct full standard disjunctive forms, then $|A|^+_e \neq |B|^+_e$.

证明.

Suppose A and B are distinct full standard disjunctive forms. Assume there is a disjunct D in A which does not occur in B. By the previous lemma, there is a verifier s of D, and hence of A, verifying only conjunctive forms that subsume D.

Suppose for reductio that s verifies B. Then s verifies some disjunct E of B. It follows that E subsumes D. Since B is full, $v(B)$ is full too. And since $lit(E) \in v(B)$ and $lit(E)$ subsumes $lit(D)$, $lit(D) \in v(B)$. Since B is standard, B includes $cf(lit(D))$ as a disjunct. But D itself is a standard conjunctive form, so $D = cf(lit(D))$ and B includes D as a disjunct, contrary to the assumption. So s does not verify B and hence $|A|^+_e \neq |B|^+_e$. □

Now we can prove the completeness theorem.

Proposition

For all equivalences $A \approx B$ in $\mathcal{L}_{\approx}$, if $\models_e A \approx B$ then $\vdash_e A \approx B$.

证明.

Prove by contraposition. Suppose $\nvdash_e A \approx B$. By the normal form theorems, A and B are provably equivalent within $\mathfrak{D}_e$ to full standard disjunctive forms A^* and B^* and $A^* \neq B^*$ (otherwise we can prove $A \approx B$). By the previous proposition, $|A^*|_e^+ \neq |B^*|_e^+$. By soundness, $|A|_e^+ = |A^*|_e^+ \neq |B^*|_e^+ = |B|_e^+$. Therefore $\not\models_e A \approx B$. □

Note that the presentation of the proof here is different from the paper.

References I

Correia, F. (2020). Granularity. In Raven, M. J., editor, *The Routledge Handbook of Metaphysical Grounding*, pages 228–243.

Fine, K. (2016). Angellic content. *Journal of Philosophical Logic*, 45(2):199–226.

Fine, K. (2018a). Compliance and command ii, imperatives and deontics. *The Review of Symbolic Logic*, 11(4):634–664.

Fine, K. (2018b). Compliance and command i—categorical imperatives. *The Review of Symbolic Logic*, 11(4):609–633.

Krämer, S. (2019). Ground-theoretic equivalence. *Synthese*, pages 1–41.

Nute, D. (1980). *Topics in Conditional Logic*, volume 20. Springer Science & Business Media, Netherlands.

郭顺利 (2012). 条件句逻辑的非标准扩充. 博士论文, 北京大学.