TMS for Modal Loigcs

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1 Truthmaker Semantics

1.1 Comparison

	Truthmaker Semantics	Possible World Semantics
Logical Space	States	Possible Worlds
	can be incomplete and/or inconsistent	complete and consistent
Truthmaking	Exact Verification	Necessitation
	fully relevant	not fully relevant
	non-monotonic	monotonic

Table 1: Differences between TMS and PWS

1.2 State Space

Definition 1 (State Space)**.** *A state space is an ordered pair $\langle S, \sqsubseteq \rangle$, where S (states) is a set and $\sqsubseteq$ (parthood/substate relation) is a complete partial order* *on S. (A order is complete iff each subset of its domain has a least upper bound w.r.t. the order.)*

Definition 2 (Fusion)**.** *Given a subset $T = \{t_1, t_2, ...\}$ of S, we call the least upper bound of T the fusion of T. We use $\sqcup T$ or more often $t_1 \sqcup t_2 \sqcup ...$ to represent the fusion of T.*

Some results:

Proposition 3. *There is a least state $\square = \bigsqcup \emptyset$ and a greatest state $\blacksquare = \bigsqcup S$ w.r.t. $\sqsubseteq$ in a state space.*

Proposition 4. *Each subset $T = \{t_1, t_2, ...\}$ of S has a greatest lower bound ($\bigsqcup\{s \in S : s \sqsubseteq t$ for all $t \in T\}$). We donote it by $\bigsqcap T$ or $t_1 \sqcap t_2 \sqcap ...$.*

1.3 Models

Definition 5 (Truthmaker Model)**.** *A truthmaker model is a triple $\langle S, \sqsubseteq, |\cdot|\rangle$ where $\langle S, \sqsubseteq\rangle$ is a state space and $|\cdot|$ a function mapping each state s to a pair $\langle |s|^+, |s|^-\rangle$ with $|s|^+$ and $|s|^-$ non-empty sets of sentential letters.*

Evaluation of boolean sentences is defined by the following clauses:

Definition 6. *(i)+ $s \Vdash p$ iff $p \in |s|^+$;*

(i)- $s \dashv\!\vert\, p$ iff $p \in |s|^-$;

(ii)+ $s \Vdash \neg B$ iff $s \dashv\!\vert\, B$;

(ii)- $s \dashv\!\vert\, \neg B$ iff $s \Vdash B$;

(iii)+ $s \Vdash B \wedge C$ iff for some t, u such that $t \Vdash B, u \Vdash C$ and $s = t \sqcup u$;

(iii)- $s \dashv\!\vert\, B \wedge C$ iff $s \dashv\!\vert\, B$ or $s \dashv\!\vert\, C$;

(iv)+ $s \Vdash B \vee C$ *iff* $s \Vdash B$ *or* $s \Vdash C$*;*

(iv)- $s \dashv\!\mid B \vee C$ *iff for some* t, u *such that* $t \dashv\!\mid B, u \dashv\!\mid C$ *and* $s = t \sqcup u$*.*

For convenience, I will use $|A|^+$ and $|A|^-$ to denote the set of all exact verifiers/falsifiers of A.

In terms of exact verification and falsification, Fine also defines two looser relations:

Definition 7. *Given a model* $\mathfrak{M}$*, for a state s and a formula A, s inexactly verifies A* ($s||\!> A$) *iff there is a* $t \sqsubseteq s$ *such that* $t \Vdash A$*; s inexactly falsifies A* ($s <\!||A$) *iff there is a* $t \sqsubseteq s$ *such that* $t \dashv\!\mid A$*.*

2 Relation to Boolean Semantics

The author suggests that Truthmaker Models also provide an "exactification" of Boolean (Truth-valuation) Semantics in the sense that all the semantical notions in the latter can be definedin the former. Take the semantics for classical logic as an example. We can resurrect the semantics in TMS by considering the special class of *atomically sound and complete* states:

Definition 8 (Atomically Sound and Complete States)**.**

1. *A state s is atomically sound if for no p,* $s||\!> p$ *and* $s <\!||p$*;*
2. *A state s is atomically complete if for any p,* $s||\!> p$ *or* $s <\!||p$*.*

Then, Γ classically entails A just in case for any truthmaker model $\langle S, \sqsubseteq, |\cdot|\rangle$ and any atomically sound and complete state $s \in S$, if $s||\!> B$ for all $B \in \Gamma$, then $s||\!> A$.

3 Informal Analysis of the Modal Operators

The analysis is based on the following "Kripke's Principle (KP)":

> $\Box A$ is true only if the apparent possibilities of A being false are not real.[1]

We may understand this as suggesting that a sentence is necessarily true if all its falsifiers are banned. So the author suggests introducing a new function into the model representing the "banning" relation. Note that the author posits a special subset of states M consisting of "modal states". Only modal states can ban other states. So for each $s \in M$, $\beta(s)$ is the set of states banned by s.

Now let's turn to the exact verifiers for $\Box A$. When A has only one falsifier, then any modal state banning this falsifier suffices to be an exact verifier for $\Box A$. But when A has multiple falsifiers, there may not be one modal states banning all falsifiers for A. In this case, we may consider multiple modal states that jointly ban all falsifiers for A. So, we define a function f, intuitively, *a ban on the exact falsifiers for* A, from $|A|^-$ to M such that for each $t \in |A|^-$, $t \in \beta(f(t))$.[2]

Having this in mind, we can understand an exact verifier for $\Box A$ as the fusion of "the bans on the exact falsifiers for A":

> $s \Vdash \Box A$ iff $s = \bigsqcup ran(f)$, where f is a ban on the exact falsifiers for A.

Let's turn to $\Diamond A$. We cannot determine what makes $\Diamond A$ true simply based on waht a modal state ban. We need to assign to each modal state s a set $\alpha(a)$ of states allowed by s. Then we have

> $s \Vdash \Diamond A$ iff there exists some $t \in |A|^+$ s.t. $t \in \alpha(s)$.

[1] Consider Kripke's argument for dualism.

[2] Intuitively, f helps us find a state ("a ban on t") that bans t.

For falsification, we can take the duals of the verification conditions:

$s \dashv \Box A$ iff there exists some $t \in |A|^-$ s.t. $t \in \alpha(s)$.

$s \dashv \Diamond A$ iff $s = \bigsqcup ran(f)$, where f is a ban on the exact verifiers for A.

Putting it together, a truthmaker semantics for modal logics can be based on the following model:

Definition 9. *A modalized truthmaker model (m-model in short) is a quadruple $\langle S, \sqsubseteq, \mu, |\cdot|\rangle$, where:*

1. *$\langle S, \sqsubseteq\rangle$ is a state space;*
2. *μ is a function with domain $M \subset S$ assigning each $s \in M$ a pair $\langle \alpha(s), \beta(s)\rangle$ of subsets of S;*
3. *$|\cdot|$ is a function mapping each $s \in S$ to a pair $\langle |s|^+, |s|^-\rangle$ of non-empty sets of sentential letters.*

4 Relation to the Kripke Semantics

It remains to show that the propose semantics exactifies the Kripke Semantics. The crux is to give an account of accessibility in terms of the allowing and banning relations. To do this, we need an inexact conception of allowing and banning:

Definition 10. *t is inexactly allowed by s ($t \in \overline{\alpha}(s)$) just in case $t \in \alpha(s')$ for some modal state $s' \sqsubseteq s$;*

t is inexactly banned by s ($t \in \overline{\beta}(s)$) just in case $t \in \beta(s')$ for some modal state $s' \sqsubseteq s$.

We would expect the inexact version of allowing and banning to be closed under certain conditions:

$\overline{\alpha}(s)$ should be *downward closed*: if $t \in \overline{\alpha}(s)$ then for all $t' \sqsubseteq t$, $t' \in \overline{\alpha}(s)$;

$\overline{\beta}(s)$ should be *upward closed*: if $t \in \overline{\beta}(s)$ then for all $t' \sqsupseteq t$, $t' \in \overline{\beta}(s)$.[3]

Then we should define possible worlds in the modalized state space.[4] A possible world should be *sound and complete* not only in the sense that it determines the truth of every atomic sentences (and hence of boolean sentences) but also in determining the truth of all modal sentences.

Definition 11. *s is modally sound iff for no t, $t \in \overline{\alpha}(s) \cap \overline{\beta}(s)$;*
s is modally complete iff for all t, $t \in \overline{\alpha}(s) \cup \overline{\beta}(s)$.

Having these in mind, we have the following equivalence between accessibility and inexact allowing:

$$w' \text{is accessible from w} \Leftrightarrow w' \in \overline{\alpha}(w)$$

Now, we would expect that the truth of a modal sentence at a world corresponds to the existence of some of its verifiers. The case for $\Box A$ is straightforward. If $\Box A$ is made true at w, than all worlds where A is false are inexactly banned by w. For $\Diamond A$, we need the following "robust condition" of w:

(R) If $t \in \overline{\alpha}(w)$, then there is a possible world w' s.t. $t \sqsubseteq w'$ and $w' \in \overline{\alpha}(w)$.

[3] We can define $t \in \overline{\beta}(s)$ just in case there are some $s' \sqsubseteq s$ and some $t' \sqsubseteq t$ s.t. $t' \in \beta(s)$. Then the closure condition is automatically satisfied.

[4] Note that Fine (2017) provides a different way to construct possible worlds in state spaces.

So, we can take a world in the proposed account to be a modally sound and complete state that is robust. With these definitions and conditions, we can show that a modal sentence is true at a world just in case there is some of its truthmaker obtaining at (in the sense of being a part of) the world. We can also understand consequence of modal logic as truth-preservation at all worlds:

For $\Gamma \cup \{A\}$ a set of modal sentences, Γ entails A iff for any m-model $\langle S, \sqsubseteq, \mu, |\cdot|\rangle$ and world $w \in S$, if $w||\!> B$ for all $B \in \Gamma$, then $w||\!> A$.

5 Models without Possible Worlds

One may be reluctant to involve possible worlds in a state-based semantics. The author proposes another way to understand modal logic in m-models.

Definition 12. *Given a m-modal $\langle S, \sqsubseteq, \mu, |\cdot|\rangle$:*

1. *A state is a modal boundary iff s is modally sound and any proper extension of s is not;*
2. *The set of absolute possibilities $S^{\diamond} = \{t \in S \mid t \sqsubseteq s$ for some modal boundary $s \in S\}$.*

Note that a modal boundary is not necessarily a world in the above sense. In fact, there are (normal) m-models without any world. See Example 4 in Chapter 2. The author suggests understanding the semantic notions in terms of absolute possibilities. For example, the consequence relation can be defined as:

For $\Gamma \cup \{A\}$ a set of modal sentences, Γ entails A iff for any m-model $\langle S, \sqsubseteq, \mu, |\cdot|\rangle$ and $s \in S^{\diamond}$, if $s||\!> B$ for all $B \in \Gamma$, then $s \not<\!||A$.

However, this may not establish normal modal logics, for the axiom K is possibly falsified by some absolute possibilities. The crux is that when a state t is not compatible with both the verifiers and falsifiers of a sentence, from the perspective of a possibility s, t must be taken as impossible by s. This can be guaranteed by the condition:

> For a modal boundary s[5], if t is *s-incompatible* with $|A|^+ \cup |A|^-$ with A some sentence, i.e.,
>
> $$\{t\} \sqcup (|A|^+ \cup |A|^-) \subseteq \overline{\alpha}(s),$$
>
> then $t \in \overline{\alpha}(s)$.

[5] It suffices to only impose the constraint on modal boundaries. See the proof of Theorem 5.

This condition can be further simplified. Models with modal boundaries satisfying these (and some further) conditions are called normal m-models.

6 From Normal M-models to W-models

W-models are normal m-models whose modal boundaries are sound and complete. It can be shown that every normal m-model can be "completed" into a w-model. See Section 2.5.

Since we can move back and forth between Kripke models and w-models (which are also normal m-models), we can show that the class of Kripke models and the class of normal m-models determine the same logic.

References

Fine, K. (2017). Truthmaker Semantics. *A Companion to the Philosophy of Language*, pages 556–577.