# "Actual and Potential Infinity" by Øystein Linnebo and Stewart Shapiro

Handout zur Struktura-Lesegruppe
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## 1 Historical Background

1.1 There is a long tradition in favor of potential infinity.

1.2 Only after the Cantorian success has the proponents of potential infinity been limited largely to intuitionists only.

1.3 However, rather than presuming a direct connection between potential infinity and intuitionistic logic, it may be the case that they have a common cause.

1.4 As is usually maintained, one common cause of the two ideas is *anti-realism*.

1.5 Alternatively, one might explore the option that the common cause lies in *indefinite extensibility* of the domain of quantification, though this option needs to be fleshed out, regarding both its semantics and its ontology (if it is to be differentiated from anti-realism).

1.6 In §2, the author quoted Aristotle:

> [...] But it is only in one sense that the magnitude is divisible through and through, viz. in so far as there is one point *anywhere* within in and all its points are *everywhere* within it if you take them singly. (317a3-8, my emphasis)

While according to Jonathan Lear's analysis invokes a kind of modality (of the generative or procedural activities of a perhaps idealized mathematician):

> Aristotle's thesis is "that the structure of the magnitude is such that any division will have to be only a partial realization of its infinite divisibility: there will have to be possible divisions that remain unactualized"

1.7 In fact, even Cantor himself had occasionally spoken of the "absolutely infinite" as "inconsistent multiplicities", given that it presumes the "all-in-one principle" (cf. Cartwright (1994)). By contrast, a 'set' is a consistent multiplicity.

## 2 Modal Explications of Potential Infinity

2.1 The infinite divisibility of a stick:

$$\Box\forall x(Pxs \to \Diamond\exists y Pyx)$$

But also: the incompletability of division:

$$\neg\Diamond\forall x(Pxs \to \exists y Pyx)$$

2.2 Similarly, regarding the infinity and incompletability of natural numbers, we have:[1]

$$\Box\forall m\Diamond\exists n \textsc{Succ}(m,n)$$

$$\neg\Diamond\forall m\exists n \textsc{Succ}(m,n)$$

2.3 Regarding the modal analysis, now two questions arise:

(i) The question regarding the logic of the modal system for potential infinity.

(ii) The question regarding the kind of translate relationship between the modal language and the non-modal language in ordinary practice of mathematics (which is arguably not fully explicit), and regarding which entailment relations obtain in the translated non-modal language.

2.4 There is also the question regarding the modal truths — its clause in translated non-modal language ( $\neq$ metalanguage: how to really understand the difference ?? ) — we can thereby identify two variants:

(v1) *Liberal Potentialism*: There are objective truths about the relevant modal aspects of reality, and this objectivity warrants the use of some classical form of modal logic. (E.g. the argument could be that the rule for decimal expansion of a real number is fixed. Eventually this could lead to classical FOL in translation.)

[1] Note that a more liberal, generalized form of potentialism might merely claim that some, but not all, actually infinite process can be completed, thus validating e.g. $\Diamond\forall x(Pxs \to \exists y Pyx)$, i.e. although it is impossible to complete the process of forming sets from any objects that are available, but any generative process that is indexed by a set-theoretic ordinal can be completed.

(v2) *Strict Potentialism*: The modal truths themselves require certain 'procedural' reading, such that not only that every object be generated at some stage of a process, but also that every truth be "made true" at some stage. (But for a quantified sentence to be "true" even before all the objects with which the sentence is concerned have been generated, the translated non-modal logic needs to be intuitionistic.)

# 3 The Modal Logic

3.1 The authors' potentialism assumes that every possible contains only finitely many objects. (In contrast, the possibility of an actual infinity will be realized at a possible world if it contains infinitely many objects.)

3.2 The authors' potentialism also assumes that objects are not destroyed in the process of construction or generation. This corresponds to the domain-expansion constraint: for all $w \in W$, if $wRw'$ then $D_w \subseteq D_{w'}$, which leads to (CBF).
Moreover, assuming that a possible world is determined completely by the (mathematical) objects it contains, one could also add that $w = w'$ if $D_w = D_{w'}$.

3.3 The authors claim that the validation of (CBF) makes the modality unlike a metaphysical modality. On the contrary, we can either:

(a) Understand it as a restriction of "ordinary" metaphysical modality. (E.g. as a modality whose accessibility relation is defined out of expanding domains)

(b) Maintain that it is an altogether distinct kind of modality.

3.4 The frame condition requires that $R$ is a partial order. (May further demand that it is well-founded.) Moreover, the authors suggest to add the condition of *convergence*, which says that the license to generate a mathematical object is never revoked as our domain expands; in other words, whenever we have a choice of *mathematical* objects to generate, the order in which we choose to proceed is irrelevant:

$$\Diamond\Box p \to \Box\Diamond p \tag{G}$$

Thus the frame condition suffices to validate system **S4.2** (T4G), plus quantifier rules in QL.

3.5 The *stability* condition:

$$\phi \to \Box\phi$$

$$\neg\phi \to \Box\neg\phi$$

echoes the heredity constraint in intuitionistic logic for atoms.

## 4 The Translation

4.1 With the modal system specified as above, it seems obvious that we can adopt Gödel's translation of propositional intuitionistic logic into **S4**.

4.2 But the authors contend that this translation is unfit for potentialists, since it would imply rejection of incompletability.

4.3 The authors proposed a *potentialist translation*, according to which all of a formula's quantifiers should be fully modalized, which not only translates $\forall$ into $\Box\forall$, *but importantly, also* $\exists$ *into* $\Diamond\exists$. But each connective is translated as itself.

4.4 With all these together, the authors are able to prove the '**classical potentialist mirroring**' of entailment relations.

4.5 This enables the 'liberal potentialists' (e.g. Aristotle) to maintain that the logic of potential infinity is classical (validating LEM, say). But they could still be differentiated from the 'actualists' in that they have certain *philosophical* (e.g. *metaphysical*) understanding of the modality at play.

4.6 Nevertheless, the authors are also able to prove the '**intuitionistic potentialist mirroring**' of entailment relations (given the intuitionistic decidability of all the atoms in the non-modal language). Based on these results, the authors hold that the thesis of potentialism can be separated from the question of whether the appropriate logic is classical or intuitionistic.

## 5 Higher-Order Logic of Potential Infinity

5.1 Beyond the philosophical differences, potentialists and actualists may also induce different logics in higher-order and plural languages.

5.2 If we apply potentialist translation to the unrestricted plural comprehension scheme, we will have:

$$\Diamond\exists x\phi(x) \to \Diamond\exists xx\Box\forall u(u \prec xx \leftrightarrow \phi(u))$$

But the authors reject this principle, because we need to reject the following

$$\Box\forall u(u \prec xx \leftrightarrow u = u)$$

With the domains expanding, universal plural membership of $xx$ will fail to hold (given that plural membership is rigid) but self-identity should remain satisfied, thus the two conditions cannot remain necessarily coextensive.

5.3 For this reason, the authors suggest that potentialists should restrict plural comprehension (unlike actualists, who in effect adopt full classical second-order PA).

5.4 By contrast, consider the potentialist translation of the second-order comprehension scheme:

$$\Diamond\exists F\Box\forall x(Fx \leftrightarrow \phi(x))$$

The authors suggest that there is no obvious reason why the liberal potentialist should wish to restrict this, since the concept $F$ need not be modally rigid. In this regard, therefore, actualism and liberal potentialism agree again.

5.5 The authors conclude that liberal potentialists, but not actualists, have the resource to differentiate the two procedures of set formation: by plurality and by concept: while there is no compelling reason to think that every concept defines a set, it is hard to resist the view that every plurality suffices to define a set (cf. Cantor's distinction earlier).

# 6 Strict potentialism

6.1 For strict potentialists, however, the generative process should be understood as a process of actual constructions, whereby mathematical objects and truths/proofs—which did not previously exist or obtain—are brought into being.
However, the price seems to be that it saddles strict potentialism with the controversial anti-realist views of traditional intuitionism. But it seems that the authors do not wish the potentialist modality to be given a purely epistemic reading.

6.2 The authors thereby argue that not every generalization is 'made true' by the totality of its instances—there are essence-based constraints on any future generation of the objects studied by mathematics.

6.3 Kleene's realizability interpretation, roughly: $e$ is a truth-maker for $\forall n\phi(n)$ just in case $e$ specifies a function that maps any numeral $\bar{n}$ to a truth-maker for the associated instance $\phi(\bar{n})$.

6.4 The realizability interpretation combined with intuitionistic–potentialist mirroring, entitles the strict potentialists to adopt an intuitionistic QL with non-anti-realist reading of the quantifiers ranging over indefinitely extensible domains, where every truth is 'made true' at some finite stage of the generative process.

6.5 An upshot for strict potentialists is that they will insist that the potential infinity of the natural numbers removes the license to anything stronger than intuitionistic quantification.

# 7 Discussions

7.1 The discussion of this paper (and its modal treatment) is similar, but not identical, to the debates of indefinite extensibility in absolute and relative generality. One difference is that the authors of this paper allow themselves to consider a possible understanding of the modality at stake as a *restriction* on metaphysical modality.

7.2 An important philosophical question regards how to understand the (absolute generality- or infinity-achieving) modality in all these debates. Consider the following options:

   (i) Understood *linguistically*, as a kind of interpretational modality. Flocke (2021) has tired to show that the linguistic interpretations are generally unsatisfactory.

   (ii) Understood *epistemically*, this seems problematic for generality relativists, and the authors of the current paper also reject it, due to concerns that it is obviously anti-realist.

   (iii) Understood *metaphysically*: although Flocke categorizes Fine (2006) as holding such a view, this option would actually require more substantive commitments regarding the modality's (meta-)ontological nature (as e.g. Flocke herself might want to undertake).

   (iv) Understood *postulationally*: the term is used in Fine (2006), which is to label that it is not a "genuine modality" (which, per Fine, includes metaphysical, natural, and normative modalities). In a sense, the modality should be both relative to the generative activities and independent from our mental state. Its nature is as elusive as the nature of mathematics.

## References

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