How to Count Structure

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1 The Automorphism Approach

SYM* A mathematical object X has more structure than a mathematical object Y iff $Aut(X) \subsetneq Aut(Y)$.

By mathematical objects, we mean mathematical structures. $Aut(X)$ is the set of all automorphisms of X.

1.1 Arguments for the Automorphism Approach

1.1.1 The Argument From Examples

Consider, for example, automorphisms of a topological space and automorphisms of its underlying set. The automorphism approach clearly captures out intuition about this example.

1.1.2 The Argument From Size

Automorphisms are structure-preserving. $Aut(X) \subsetneq Aut(Y)$ thus suggests that X has more structures than Y. This is because the more structures a mathematical object has, the harder for a map to be automorphic— more structures have to be preserved by a map.

1.1.3 The Argument From Definability

A natural way to interpret 'X has more structures than Y' is the following:

Desideratum X has more structure than Y iff X can define all of the structures that Y has, but X has some piece of structure that Y does not define.

The formal setup for this argument consists two signature Σ_1 and Σ_2. Assume that Σ_1-structure A and Σ_2-structure B have the same underlying set. The '*basic* structures' of A and B can be thought of as *represented* by elements of Σ_1 and Σ_2 respectively. This setup is intuitive given that the objects we tried to compare are mathematical structures (e.g. a topological space and its underlying set). A notational setup is to denote that a sequence of elements $a_1, \ldots, a_n \in A$ satisfy $\phi(x_1, \ldots, x_n)$ as $A \models \phi[a_1, \ldots, a_n]$.

Definition 1. *(Explicit Definition) A Σ_1-structure A explicitly defines p^B if there is a Σ_1-formula ϕ such that $\phi^A = p^B$.*

Definition 2. *(Implicit Definition) A Σ_1-structure A implicitly defines p^B if $h[p^B] = p^B$ for every automorphism of A.*

Explicit definitions show that every structure B has can be constructed from or is an abbreviation of ϕ^A— a piece of structure that A has.

To say that A implicit defines p^B is to claim that p^B is 'invariant under' or 'preserved by' the symmetries of A. Since symmetries reveal the invariant structures, p^B is thus a piece of structure of A

Proposition 1. *The following are equivalent:*

1. For every symbol $p \in \Sigma_2$, A implicitly defines p^B, but there is a $q \in \Sigma_1$ such that B does not implicitly define q^A.

2. $Aut(A) \subsetneq Aut(B)$.

Proof. For every $h \in Aut(A)$, $h[p^B] = p^B$ for every $p \in \Sigma_2$, then h is also an automorphism of B since it respects all $p \in \Sigma_2$. Since there is a $f \in Aut(B)$ such that $f[q^A] \neq q^A$ for some $q \in \Sigma_1$, $f \notin Aut(A)$

Since every $h \in Aut(A)$ is also in $Aut(B)$, $h[p^B] = p^B$. Let $f \in Aut(B) \setminus Aut(A)$. Since $f \notin Aut(A)$, there is a $q \in \Sigma_1$ such that $f(q^A) \neq q^A$. □

N.B. The proof relies on the fact that A and B have the same underlying set. Also, $h(\langle a_1, \ldots, a_n \rangle) = \langle h(a_1), \ldots, h(a_n) \rangle$.

Proposition 1 shows that the automorphism approach satisfies **Desideratum** when **Desideratum** is interpreted by the implicit definition.

1.2 Problems

1.2.1 Sensitivity

SYM* cannot deal with objects with different underlying sets. For example, a topological space (X, τ) and a set Y such that $X \neq Y$.

1.2.2 Triviality

SYM* is implausible when considering objects whose automorphism is the identity map (i.e. has a trivial automorphism group).

Consider the following example: Let $\Sigma_1 = \{c_1, c_2, \ldots\}$ containing countably infinite constants. Let $\Sigma_2 = \Sigma_1 \cup \{p\}$, where p is a unary predicate symbol. We let Σ_1-structure and Σ_2-structure A and B both have the domain $\{0, 1, 2, \ldots\}$ and let $c_i^A = c_i^B = i$ for each i. But the only automorphism of A is the identity map. So it cannot be the case that $Aut(B)$ is a proper subset of $Aut(A)$ although B has more structures than A in some sense.

2 The Category Approach

Definition 3. *A functor $F : \mathscr{C} \to \mathscr{D}$ is full if for all c_1, c_2 in $\mathscr{C}$ and arrow $g; Fc_1 \to Fc_2$ in $\mathscr{D}$, there exists an arrow $f : c_1 \to c_2$ in $\mathscr{C}$ with $Ff = g$.*

In other words, for all c_1, c_2 in $\mathscr{C}$, $F_{arrow} : Hom_{\mathscr{C}}(c_1, c_2) \to Hom_{\mathscr{D}}(Fc_1, Fc_2)$ is surjective.

Definition 4. *(Structure-Forgetful Functor) F **forgets structure** if F is not full.*

Note the difference between functors which forgets structures and properties. For example, the inclusion functor **Ab** $\to$ **Grp**, which is fully faithful, forgets properties but not structure, whereas **Ab** $\to$ **Set** forgets structures.

To apply the categorial method to first-order theories, we need the categories of first-order theories.

Definition 5. *(Elementary Embedding) If Σ is a signature and M, N are Σ-structures, then an elementary embedding $f : M \to N$ is a function from M to N such that $M \vDash \phi[a_1, \dots, a_n]$ iff $N \vDash \phi[f(a_1), \dots, f(a_n)]$ for any first-order formula ϕ.*

Definition 6. *$Mod(T)$ is category whose objects are models of T and whose arrows are elementary embeddings between two such models.*

2.1 Arguments for the Category Approach

2.1.1 The Argument From Examples

Consider, for example, the functor from **Top** to **Set**.

2.1.2 The Argument From Size

Consider, again, the functor U : **Top** $\to$ **Set**. Since U is not full, some arrows in **Set** are 'forgotten' by U. So there are more set functions than continuous maps. Since there are more structure-preserving maps between sets than between topological spaces, **Set** must have less structure to be preserved by these maps.

The essential idea of The Category Approach is the same as **SYM***: 'a larger collection of arrows in a category should indicate that the objects in the category have less structure.'

An interesting case to consider: A functor $F : \mathscr{C} \times \mathscr{D} \to \mathscr{C}$. More specifically, consider the category **1**, which contains only one object (call it $\bullet$) and only the identity arrow. Consider the functor G : **Set** $\times$ **1** $\to$ **Set**. We can think of **Set** $\times$ **1** as consisting of objects $\langle a, \bullet \rangle$ (a is a set) and arrows $\langle f, 1_\bullet \rangle$ (f is a set function) which is uniquely determined by f and $1_\bullet$. It seems that one might feel that G is also forgetful in some sense. Since • isn't defined on any sets, what we want to talk about seems really to be a *mathematical structure* rather than an *object* like a pair of the form $\langle a, \bullet \rangle$.

Disclaimer: I think that this point has been more detailed discussed in Baez, J., Bartels, T., Dolan, J., & Corfield, D. (2006). Property, Structure and Stuff, which I haven't read.

2.1.3 The Argument From Definability

To argue for the Category Approach from definability, we aim for a revised version of **Desideratum**:

A structure-forgetful functor $F : Mod(T_2) \to Mod(T_1)$ indicates that T_2 defines all of the structures of T_1, but T_1 posits some piece of structure that T_1 does not define.

Formal Setup:

Definition 7. *(Reconstrual) A reconstrual F of (a signature) Σ_1 into (a signature) Σ_2 is map from the elements of Σ_1 to Σ_2-formulas that take an n-ary predicate symbol $p \in \Sigma_1$ to a Σ_2-formula $Fp(x_1, \ldots, x_n)$.*

A reconstrual can be extended to a map from Σ_1-formulas to Σ_2-formulas in a natural way: we define the Σ_2-formula $F\phi(x_1, \ldots, x_n)$ as follows.

1. If $\phi(x_1, \ldots, x_n)$ is $x_i = x_j$, then $F\phi(x_1, \ldots, x_n)$ is the Σ_2-formula $x_1 = x_j$.

2. If $\phi(x_1, \ldots, x_n)$ is $p(x_1, \ldots, x_n)$, then $F\phi(x_1, \ldots, x_n)$ is $Fp(x_1, \ldots, x_n)$.

3. F commutes with Boolean connectives and quantifiers.

We still cll the map between from Σ_1-formulas to Σ_2-formulas 'reconstrual'.
Definition 8. *(Translation) Let T_1 and T_2 be theories in Σ_1 and Σ_2 respectively, a reconstrual $F; \Sigma_1 \to \Sigma_2$ is a translation $F : T_1 \to T_2$ if*

$$T_1 \models \phi \Rightarrow T_2 \models F\phi$$

.

A translation F gives rise to a map $F^* : Mod(T_2) \to Mod(T_1)$. We can construct our functor F^* in the following way: For every model A of T_2 we define $F^*(A)$ as follows:

1. $dom(F^*(A)) = dom(A)$.

2.$(a_1, \ldots, a_n) \in p^{F^*(A)}$ iff $A \models Fp[a_1, \ldots, a_n]$.

3. Obvious mappings on elementary embeddings.

Then one can show that

Lemma 1. *Let M be a model of T_2 and $\phi(x_1, \ldots, x_n)$ a Σ_1-formula. Then $M \models F\phi[a_1, \ldots, a_n]$ iff $F^*(M) \models \phi[a_1, \ldots, a_n]$.*

Definition 9. *(Essentially Surjective) A translation $F : T_1 \to T_2$ is essentially surjective if for every Σ_2-formula ψ there is a Σ_1-formula ϕ such that*

$$T_2 \models \forall x_1 \ldots \forall x_n(\psi(x_1, \ldots, x_n) \leftrightarrow F\phi(x_1, \ldots, x_n))$$

The existence of eso translation $F : T_1 \to T_2$ captures in a sense in which T_1 can define all the structures of T_2, since any formula ψ in the language of T_2 is expressible using the language of T_1. The eso of F guarantees that there is some formula ϕ in the language of T_1 that translates to (a logical equivalent of) ψ.

Assuming that Σ_1 an $Sigma_2$ contain only predicate symbols and are disjoint, we have

Proposition 2. *Let F be a translation $F : T_1 \to T_2$. The following are equivalent:*

1. F is essentially surjective.

2. $F^* : Mod(T_2) \to Mod(T_1)$ is full.

This proposition satisfies our revised desideratum: Suppose that F^* is not full. (1) Since F^* is induced by a translation $F : T_1 \to T_2$, there is a sense in which T_2 can define all of the structures of T_1. For each piece of structure p that T_1 posits, T_2 posits Fp, and we can therefore use this piece of structure to define p. So T_2 defines all of the structure of T_1. (2) Since F^* is not full, Proposition 2 guarantees that F is eso, and so there is a formula ψ in the language of T_2— in other words, a piece of structure that T_2 posits— for which there is no corresponding piece of structure ϕ posited by T_1 that F translates to ψ. This means that T_1 does not define all of the structures of T_2 But if F^* is full, T_1 does define all of the structure of T_2.

2.2 Problems

2.2.1 Sensitivity and Triviality

Sensitivity is clearly avoided. Consider the example we considered previous about triviality. The Category Approach offers a correct verdict.

Let $Th(B)$ be the $\Sigma_1 \cup \{p\}$-theory that has as axioms every $\Sigma_1 \cup \{p\}$-sentence ϕ such that $B \models \phi$, and let $Th(A)$ be the Σ_1-theory that has as axioms every Σ_1-sentence ψ such that $A \models \psi$. Consider the translation $F : Th(A) \to Th(B)$ defined by $F : c_i \mapsto c_i$ for every $c_i \in \Sigma$. It's clear that F is a translation since if $A \models \phi$, then $B \models \phi$. But since it cannot be the case that $Th(B) \models \forall x(\phi(x) \leftrightarrow p(x))$, F is not eso. According to Proposition 2, $F^* : Mod(Th(B)) \to Mod(Th(A))$ is not full.

2.2.2 Relativization to the Functor

There are usually many functors between two categories, but our category approach depends on the choice of functors.

Consider $\Sigma_1 = \{p, q\}$ and $\Sigma_2 = \{r, s\}$, where all these symbols are unary. Let Σ_1-theory T_1 and Σ_2-theory T_2 be empty theories. Consider the following three translations:

$$F : p \mapsto r, q \mapsto s$$

$$G : p \mapsto r, q \mapsto r$$

$$H : r \mapsto p, s \mapsto p$$

F is eso, G and H are not. So F^* is full but G and H are not. Which functor/translation should we pick? As Barrett claims, it is often easy to choose a functor between physical theories. But what about other cases? One might be tempted to say that we should choose the most 'charitable' one in the example above. But how far can we go with this?

3 Two Payoffs

3.1 Excising A Piece of Structure?

Let $\Sigma = \{p, r\}$ and T be a Σ-theory with the one axiom $\forall x(r(x) \leftrightarrow p(x,x))$. Consider the $\{p\}$-theory T^- with no axioms. It seems that p is excised since it doesn't even appear in any sentences T^- entails. But it seems that no structures are excised since r is definable from p (especially if we think that symmetry tells us the structure of a theory). It is easy to check that M of T has the same automorphism group as $M|_{\{p\}}$ of T^-. If we consider $F : T^- \to T$ that maps p to itself, then it is also easy to check that it is eso. Proposition 2 therefore says that these theories have the same structures.

The moral to be drawn here is that excising a piece of structure is not just reformulating the theory in such a way that the piece of structure is no longer explicitly appealed to.

3.1.1 Equivalence

The fact that two theories explicitly appeal to different collections of structures in their formulations does not imply that they are inequivalent.