Notes on Bacon's "Counterfactuals, Infinity and Paradox"

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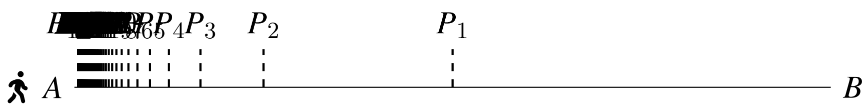
1 Paradoxes of Infinity

I take Berardete's Paradox as an example:



Some desiderata:

Divine Disposition $P_{\leq n+1} \boxright \neg P_n$

Anti-Zenoism $\neg(\bigvee_n P_{\leq n} \boxright \neg \bigvee_n P_{\leq n})$

Principles of counterfactual logic:

Identity $\vdash A \boxright A$

Substitution $A \boxright B \vdash A' \boxright B$ when A and A' are classically equivalent.[10]

Weakening $A \boxright B \vdash A \boxright B'$ when B entails B'.

Disjunction $A \boxright C, B \boxright C \vdash A \vee B \boxright C$

Infinite Conjunction $A \boxright B_1, A \boxright B_2, ... \vdash A \boxright \bigwedge_n B_n$

An inconsistency result:

Theorem 2.1. *Divine Dispositions and Anti-Zenoism are inconsistent with the principles of counterfactual logic listed in section 1.*

We argue similarly:

1. $P_{\geq 2} \mathrel{\Box\!\!\to} \neg P_{\geq 1}$ (from Dispositions by Weakening)
2. $(P_{\geq 1} \vee P_{\geq 2}) \mathrel{\Box\!\!\to} \neg P_{\geq 1}$ (from 1 by Substitution)
3. $(\neg P_{\geq 1} \wedge (P_{\geq 3} \vee P_{\geq 4} ...)) \mathrel{\Box\!\!\to} \neg P_{\geq 1}$ (by Entailment)
4. $(P_{\geq 1} \vee P_{\geq 2} \vee (\neg P_{\geq 1} \wedge (P_{\geq 3} \vee P_{\geq 4} ...))) \mathrel{\Box\!\!\to} \neg P_{\geq 1}$ (from 2 and 3 by Disjunction)
5. $(P_{\geq 1} \vee P_{\geq 2} \vee ...) \mathrel{\Box\!\!\to} \neg P_{\geq 1}$ (by Substitution)

As before, we may generalize the above reasoning to derive $(P_{\geq 1} \vee P_{\geq 2} \vee ...) \mathrel{\Box\!\!\to} \neg P_{\geq n}$ for every n, and so by Infinite Conjunction and Weakening contradict Anti-Zenoism.[16]

where Entailment is: $\vdash A \mathrel{\Box\!\!\to} A'$ whenever A classically entails A'.

2 Revising Counterfactual Logic

2.1 Infinite Conjunction

2.2 Substitution

Fine (2012) discussed a similar paradox of infinity involving counterfactuals, and proposed a semantics for counterfactuals which invalidates Substitution.

His semantics is based on *state space*, which consists of incomplete *states*. While classically equivalent sentences are true in the same possible worlds, they need not be verified by the same states.

Fine's clause for counterfactuals is:

> $A \mathrel{\Box\!\!\to} C$ is true at w iff for every states t that exactly verifies A, C is inexactly verified by every u such that $t \to_w u$ (u is a possible outcome of imposing t on w).

Bacon's Criticism I Fine offers a model theory but leave unsettled many questions of truth.

Fine's Reply We can have a causal or interventionist reading of the relation $\rightarrow$. See Briggs (2012).

Comment I I think the difference between Fine (2012)'s transition relation and more classic slection function is less substantive than people thought. In classic selection function models, we are looking at $f(A, w)$ (where A can be a proposition or a sentence), the selected worlds in which A is true. Fine's transition relation can be re-written as $u \in f(t, w)$. So to evaluate a counterfactual, we are considering $f(t, w)$, the relevant set of states in which t, a verifier of the antecedent, obtains. So, many interpretations of the classic select function can be applied to Fine's model theory. The question is whether the interpretation is compatible with the hyperintensional model theory.

One distinguished feature of Fine's semantics is that it validates the rule of Simplicifation of Disjunctive Antecedent (what he calls Simplification)[1]:

> **Simplification** $A \vee B \mathrel{\Box\!\!\rightarrow} C \vdash A(B) \mathrel{\Box\!\!\rightarrow} C$

It is known that Simplification, together with Substitution, gives rise to the notorious Antecedent Strengthening:

> **Antecedent Strengthening** $A \mathrel{\Box\!\!\rightarrow} C \vdash A \wedge B \mathrel{\Box\!\!\rightarrow} C$

Since Fine's semantics invalidates Substitution, it's safe to have Simplification without Antecedent Strengthening.

Bacon's Criticism II Even if some hyperintensional semantics is safe from Antecedent Strengthening, it still validates another unfavorable rule, given a limited version of substitution:

[1]In fact his semantics validates the stronger equivalence between $A \vee B \mathrel{\Box\!\!\rightarrow} C$ and $(A \mathrel{\Box\!\!\rightarrow} C) \wedge (B \mathrel{\Box\!\!\rightarrow} C) \wedge (A \wedge B \mathrel{\Box\!\!\rightarrow} C)$. But what is important here is just Simplification

Weak Antecedent Strengthening $A \boxright C, B \boxright C \vdash (A \wedge B) \boxright C$

Unwelcome, since we should affirm the first and second counterfactual, but not the third.

- ✓ If I were to drink this hot beverage, I'd have a pleasant time.
- ✓ If I were to ride this roller-coaster, I'd have a pleasant time.
- ✗ If I were to drink this hot beverage and ride this roller-coaster, I'd have a pleasant time.

A hyperintensional semantics, though invalidating the full version of Substitution, must allow some limited version. For example, Fine's semantics permits these instances of substitution:

Idempotence $(A \wedge A) \boxright C \dashv\vdash A \boxright C \dashv\vdash (A \vee A) \boxright C$

Commutivity $(A \vee B) \boxright C \dashv\vdash (B \vee A) \boxright C$

Distributivity $A \wedge (B \vee C) \boxright D \dashv\vdash ((A \wedge B) \vee (A \wedge C)) \boxright D$

Associativity $(A \vee B) \vee C \boxright D \dashv\vdash A \vee (B \vee C) \boxright D$

Moreover, if the semantics validates the following equivalence:

The Simple Account $(A \vee B) \boxright C \dashv\vdash (A \boxright C) \wedge (B \boxright C)$

the unfavorable result follows:

No-Go Theorem Every theory of conditionals that permits the four limited applications of Substitution listed and validates "The Simple Account", also contains Weak Antecedent Strengthening.

The derivation of Weak Antecedent Strengthening from these assumptions goes as follows:

1. $A \boxright C$ (assumption)
2. $B \boxright C$ (assumption)
3. $A \vee B \boxright C$ (Simple Theory)
4. $(A \vee B) \wedge (A \vee B) \boxright C$ (Idempotence)
5. $((A \vee B) \wedge A) \vee ((A \vee B) \wedge B) \boxright C$ (Distributivity)
6. $(A \wedge (A \vee B)) \vee (B \wedge (A \vee B)) \boxright C$ (Commutativity)
7. $((A \wedge A) \vee (A \wedge B)) \vee ((B \wedge A) \vee (B \wedge B)) \boxright C$ (Distributivity)
8. $(A \vee (A \wedge B)) \vee ((B \wedge A) \vee B) \boxright C$ (Idempotence)
9. $(A \vee B) \vee ((A \wedge B) \vee (A \wedge B))) \boxright C$ (Associativity, Commutivity)
10. $(A \vee B) \vee (A \wedge B) \boxright C$ (Idempotence)
11. $A \wedge B \boxright C$ (Simple Theory)

Note that there is a problem in Bacon's formulation of the theorem: the instances of Substitution allows substituing the antecedent *as a whole*, but in the derivation, he substitutes a *subformula* of the antecedent with its equivalent. However, we can prove that this is permittable in Fine's semantics.

There are some ways out: Fine (2012) invalidates the Simple Theory by limiting the right-to-left direction: it is valid only in case that A and B are logically incompatible. Bacon thinks that this treatment is a little unprincipled.

The second option, which is advocated by Fine in some recent versions truthmaker semantics (though not for counterfactuals), is to differentiate A and $A \wedge A$. This option, as Bacon argues, is too radical.

Despite of the plausibility of counterexamples to Substitution, there are pragmatic strategies to explain away these counterexamples. Moreover, we have positive reasons to save Substitution in counterfactual contexts. Counterfactuals are known to be related to many metaphysical concepts, which are hardly hyperintensional.

Fine's Reply II Now Fine is more willing to give up Conjunctive Idempotence and accept the unrestricted form of Disjunction. Fine, however, does not think that the equivalence of A and $A \wedge A$ in counterfactual contexts is unquestionable.

> But what if we try to control for this by stating the two conjuncts in slightly different form? Suppose that someone has ingested some poison but not taken any antidote, and suppose that if he had taken a single dose of antidote he would have had a grisly death but not if he had taken two doses of the antidote. Then the counterfactual "if at some time he had taken a dose of the antidote, he would have had a grisly death" is true, while the counterfactual "if at some time he had taken a dose of the antidote and at some time, the same or different, had taken a dose of the antidote, then he would have had a grisly death" may well not be true, since the antecedent leaves open the possibility of his having taken two doses of the antidote.

Moreover, Fine emphasizes that the difference between A and $A \wedge A$ is due to the difference in their *roles* as antecedents. According to Fine, the role a sentence plays as an antecedent is determined by the ways in which it is *realized*, i.e., their exact truthmakers. If A has two distinct truthmakers s and t, $s \sqcup t$ will be a truthmaker for $A \wedge A$ but not for A.

The failure of substitution of classical equivalence has been thought to be characteristic of representational contexts, as Bacon believes. Fine argues that, however, the hyperintensionality involved in counterfactuals concerns the difference in ways in which the world can realize the content.

However, giving up Conjuntive Idempotence leaves us an asymmetry between conjunction and disjunction, which is rather unnatural. Moreover, Fine recognized that there is a logical reason to give up Disjunctive Idempotence with the Conjunctive one:

It now strikes me, however, that there are good reasons for not wanting A to be antecedently equivalent to (A ∨ A).[7] The issue arises from not merely wanting (A > C) ∧ (B > C) and (A ∨ B) > C to be *truth-conditional* equivalents in the sense of always having the same truth-value but also to be *exact* (*positive*) equivalents in the sense of always having the same (exact) truth-makers. Let ⊤ be the trivial truth, made true by the null state □ alone. Then we may also want A and ⊤ > A to be exact equivalents. Suppose now that ⊤ and ⊤ ∨ ⊤ were exact equivalents. Then the following sequence of formulas would all be exact equivalents: A, ⊤ > A, ⊤ ∨ ⊤ > A, (⊤ > A) ∧ (⊤ > A), A ∧ A. This suggests that, in order to avoid the exact equivalence

This position requires a modification to the semantics, one in which the positive content of A ∨ B is not simply given by the truthmakers for A and for B. I have proposed in this connection that we also take into account how *often* a given state is a truthmaker. Thus ⊤ and ⊤ ∨ ⊤ will have the same truth-maker, viz. the null state □, but ⊤ will have it as a truthmaker once, while ⊤ ∨ ⊤ will have it twice; and, in general, if the state *s* is m times a truthmaker for A and n times a truthmaker for B, then it will be m + n times a truthmaker for A ∨ B.

Comment II As Fine's reply shows, while Bacon's No-Go theorem can be used as evidence to argue against a hyperintensional account of counterfactuals, the proponents of the hyperintensionality of counterfactuals can also use this as a logical reason to reject (at least one of) Idempotences in counterfactual contexts. Moreover, while the Simple Account is one reason for hyperintensional counterfactuals, it is not the *only* reason. Other reasons include nonvacuous counterpossible conditionals (See Berto et al. (2018)). No-Go theorem is not a conclusive objection to a hyperintensional theory.

As for the derivation from Disjunctive Idempotence to Conjuctive Idempotence, I think the requirement of *exact equivalence* between $A \vee B \mathrel{\Box\!\!\rightarrow} C$ and $(A \mathrel{\Box\!\!\rightarrow} C) \wedge (B \mathrel{\Box\!\!\rightarrow} C)$ is too demanding. Note that one idea behind Fine's semantics for conditionals is the the principle of *Universal Realizability of the Antecedent*: a verifier for the counterfactual must function to lead *every* verifier for the antecedent to *some* for the consequent. However, consider $\top \vee \top \mathrel{\Box\!\!\rightarrow} A$ and $(\top \mathrel{\Box\!\!\rightarrow} A) \wedge (\top \mathrel{\Box\!\!\rightarrow} A)$, where A has two distinct verifiers s and t:

$$|\top \vee \top \mathrel{\Box\!\!\rightarrow} A|^+ = \{s, t\} \qquad |(\top \mathrel{\Box\!\!\rightarrow} A) \wedge (\top \mathrel{\Box\!\!\rightarrow} B)|^+ = \{s, t, s \sqcup t\}$$

While $\top \vee \top \mathrel{\Box\!\!\rightarrow} A$ exactly entails $(\top \mathrel{\Box\!\!\rightarrow} A) \wedge (\top \mathrel{\Box\!\!\rightarrow} A)$, the latter only *inexactly* entails the former.

Of course, one might prefer a symmetry between conjunction and disjunction for other reasons, but Fine's distinction based on times a state is a truthmaker is somehow less appealing for me. (Note that, however, Krämer (2018, 2021)'s Mode-ified truthmaker account of grounding somehow follows this idea.) One better option, as far as I see, is Jago (2020, ming)'s disjunctive structure.

2.3 Disjunction

Bacon suggests instead that it is Disjunction, rather than Substitution, that is to be blamed. He first note that there are two main motivations for Disjunction. The first one is some linguistic data:

> 1. If John went to the party, everyone would have a good time.
> 2. If Mary went to the party, everyone would have a good time.
> 3. If John or Mary went to the party, everyone would have a good time.
>
> The example in 1-3 is representative of the sort of cases one might use to motivate Disjunction. However, one must take care not to overgeneralize from particular examples: even an invalid inference may have instances in which the premises necessitate the conclusion. And even the judgment of validity in this instance is fragile. For example, suppose that both John and Mary are the life of any party, but are hostile exes. Because they wish to avoid each other, neither intend to come to the party. Because Mary isn't intending to go to the party, it would have been the case, had John gone, that everyone would have had a good time. Similarly, because John isn't intending to come to the party, had Mary gone, everyone would have had a good time. But had Mary or John gone to the party, they might have both gone. In which case, the party would have been a disaster. So we should reject 3, despite accepting 1 and 2. The intuition against 3 can be made more vivid, if we consider the logically equivalent counterfactual
>
> $3'$. If either John or Mary or both went to the party, everyone would have had a good time.
>
> 3 and 3' are equivalent given Limited Substitution, which even the Finean semantics permits.[47] (To see this equivalence, note that steps 4-10 of our above derivation of Weak Antecedent Strengthening are reversible.)

The second reason for Disjunction is due to the dominant similarity/closeness analysis of counterfactuals. For roughly, if the closest A worlds are C worlds, and the closest B worlds are C worlds, then the closest $A \wedge B$ worlds are C worlds. A characteristic rule of ordering semantics is:

Counterfactual Equivalence (CSO) $A \Box\!\!\rightarrow B, B \Box\!\!\rightarrow A, A \Box\!\!\rightarrow C \vdash B \Box\!\!\rightarrow C$

Bacon invites us to consider the scenario

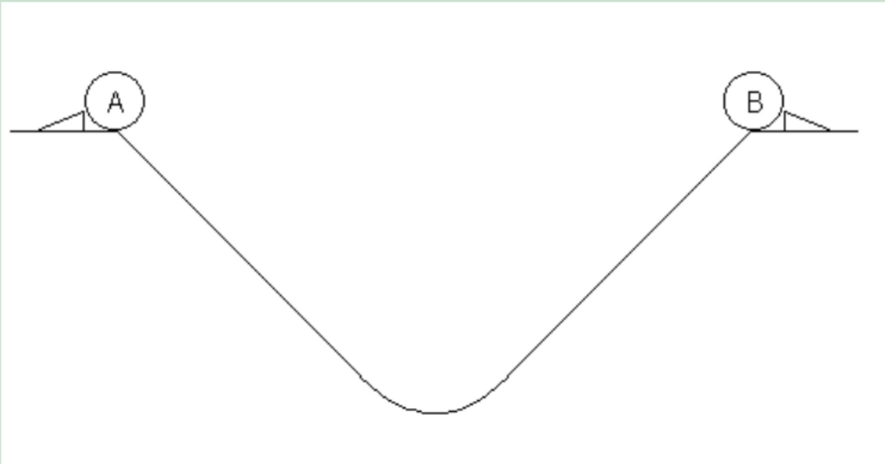


Figure 2: Two balls A and B balanced on opposing slopes.

✓ If A were to topple, the buzzer would go off.

and deny

✗ If B were to topple, the buzzer would go off.

We also have:

✓ If A were to topple, B would topple.

✓ If B were to topple, A would topple.

Moreover, Bacon points out that Fine's semantics also validates a rule that is central to the ordering analysis:

Restricted Transitivity $A \Box\!\!\rightarrow B, A \wedge B \Box\!\!\rightarrow C \vdash A \Box\!\!\rightarrow C$

But we can also find counterexample to this rule from the above scenario:

✓ If A were to topple, then B would topple.

✓ If A and B were to topple, A and B would collide somewhere in the middle.

✗ If A were to topple, A and B would collide somewhere in the middle.

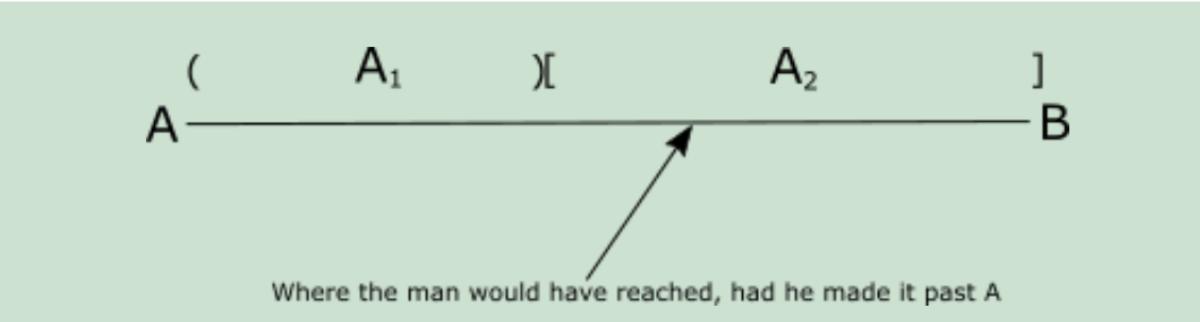
Bacon points out that Fine's argument blamed the possible worlds aspect of the classical semantics, but it is the ordering aspect that is responsible for the paradox.

Fine's Reply There is no *logical inconsistency* in adopting Restricted Transitivity.

As for the conterexample, Fine suggests revealing the underlying time parameter.

3 Bacon's Account

The idea behind the account: What would have happened if the man had made it *some* non-zero distance past A? There is a particular point, but it is a chancy matter which point it is!



Bacon's line of reasoning (as I understand): Let x be the point the man would have reached had he made it past A, and n be the greatest natural number such that $\frac{1}{2}^n$ is before x. Let A_1 be the proposition that he stopped in $(0, \frac{1}{2}^n)$, A_2 be that he stopped in $[\frac{1}{2}^n, 1]$, and C be that he stopped in $(0, \frac{1}{2}^n]$. He claims that in this scenario, we have $A_1 \mathrel{\Box\!\!\rightarrow} C$ and $A_2 \mathrel{\Box\!\!\rightarrow} C$ but not $A_1 \vee A_2 \mathrel{\Box\!\!\rightarrow} C$.

Question Why do we have $A_2 \mathrel{\Box\!\!\rightarrow} C$? Is this an echo of closeness intuition? Note that Divine Disposition claims that if he had passed $\frac{1}{2}^{n+1}$, he would be stopped at $\frac{1}{2}^n$. But the antecedent is different from A_2.

Bacon's logic of counterfactuals:

Necessitation If $\vdash B$ then $\vdash A \mathrel{\Box\!\!\rightarrow} B$

Substitution If $\vdash A \equiv B$ then $\vdash (A \mathrel{\Box\!\!\rightarrow} C) \equiv (B \mathrel{\Box\!\!\rightarrow} C)$

Normality $(A \mathrel{\Box\!\!\rightarrow} (B \supset C)) \supset ((A \mathrel{\Box\!\!\rightarrow} B) \supset (A \mathrel{\Box\!\!\rightarrow} C))$

Identity $A \mathrel{\Box\!\!\rightarrow} A$

Modus Ponens $(A \mathrel{\Box\!\!\rightarrow} B) \supset (A \supset B)$

Conditional Excluded Middle $(A \mathrel{\Box\!\!\rightarrow} B) \vee (A \mathrel{\Box\!\!\rightarrow} \neg B)$

Absurdity $(A \mathrel{\Box\!\!\rightarrow} B) \supset ((B \mathrel{\Box\!\!\rightarrow} \bot) \supset (A \mathrel{\Box\!\!\rightarrow} \bot))$

I will call this logic LC. LC can be extended to an infinitary logic, and further principles like Infinite Conjunction may be added. Since issues of completeness become more complex, I will not investigate these systems thoroughly here.

His semantics is just a classical proposition-based selection function semantics that satisfies the conditions corresponding to the axioms and rules.

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